THE (ϕ, s) REGULAR SUBSETS OF n-SPACE[1]

BY
JOHN M. MARSTRAND

1. **Notation and definitions.** Except for a special notation introduced in §5, the following is a complete summary. Some of the notation coincides with that used in Federer [2].

1.1. *Points.* Euclidean n-space, denoted by E_n, is the set of points $x = (x^{(1)}, \cdots, x^{(n)})$. The origin is 0. Further notation is $x \cdot y = x^{(1)}y^{(1)} + \cdots + x^{(n)}y^{(n)}$, $\lambda x = (\lambda x^{(1)}, \cdots, \lambda x^{(n)})$, $x - y = (x^{(1)} - y^{(1)}, \cdots, x^{(n)} - y^{(n)})$, $\rho(x, y) = |x - y|$ is the distance from x to y, $\Delta(x_1, \cdots, x_n)$ is the determinant of

$$\begin{bmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ \vdots & & \vdots \\ x_1^{(n)} & \cdots & x_n^{(n)} \end{bmatrix}.$$

A *direction* is a point θ such that $\theta^2 = 1$.

1.2. *Matrices.* The group of orthogonal $n \times n$ matrices is $\mathscr{G}_n$. The unit $n \times n$ matrix is I_n.

If

$$R = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

and x is the point $(x^{(1)}, \cdots, x^{(n)})$, then Rx is the point $(\sum_{j=1}^n a_{1j}x^{(j)}, \cdots, \sum_{j=1}^n a_{nj}x^{(j)})$. That is, when points are regarded as matrices, they are regarded as columns instead of rows.

1.3. *Sets* are denoted by capital Roman letters, but such letters sometime have other applications.

The class of Borel sets in E_n is $\mathscr{B}_n$.

Let $a \in E_n$, $R \in \mathscr{G}_n$, let k be an integer with $0 \leqq k \leqq n$, and let $\lambda > 0$. Then by

$$L_n^k(a, R, \lambda)$$

we denote the set containing all points x such that if $R(x - a) = y$, then $|y^{(i)}| \leqq \lambda$ for $i = k + 1, \cdots, n$.

Received by the editors June 5, 1963.

(1) This work was supported in part by a grant from the National Science Foundation.

Also, $M_n^k(a,R,\lambda) = E_n - L_n^k(a,R,\lambda)$.

Let $r > 0$. Then $K_n(a,r)$ is the open sphere containing all points x with $\rho(a,x) < r$, and $C_n(a,r)$ is the closed sphere given by $\rho(a,x) \leqq r$.

Further,

$$L_n^k(a,R,\lambda,r) = L_n^k(a,R,\lambda) \cap C_n(a,r)$$

and

$$M_n^k(a,R,\lambda,r) = M_n^k(a,R,\lambda) \cap C_n(a,r).$$

When $k = n-1$, the sets depend only on the last row of R, say $(\theta^{(1)},\cdots,\theta^{(n)}) = \theta$ which is a direction. Then we will sometimes use an alternative notation given by

$$L_n^{n-1}(a,R,\lambda) = L_n(a,\theta,\lambda),$$

and similarly we modify the other notation by replacing R by θ and omitting k. When θ is a direction we denote by

$$H_n(a,\theta,\lambda)$$

the set of points x such that

$$(x-a)\cdot\theta > \lambda.$$

Thus $H_n(a,\theta,\lambda) \cup H_n(a,-\theta,\lambda) = M_n(a,\theta,\lambda)$.

Again, $H_n(a,\theta,\lambda,r) = H_n(a,\theta,\lambda) \cap C_n(a,r)$.

Given a set $A \subset E_n$, we denote by $P_n^k(E)$ the (projected) set of all $(x_1,\cdots,x_k) \in E_k$ such that $(x_1,\cdots,x_n) \in A$.

We denote by $\rho(a,A)$ the distance of x to A, that is, the lower bound of $\rho(a,x)$ for $x \in A$.

Given $\lambda > 0$, λA denotes the set of points λa such that $a \in A$.

By $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ we denote respectively the closure and interior of A, and $A \times B$ and $a \times B$ denote Cartesian product sets.

1.4. *Measures.* We denote by U_n the class of all measures over E_n. That is, $\phi \in U_n$ means that

(i) $0 \leqq \phi(A) \leqq \infty$ whenever $A \subset E_n$,

(ii) the ϕ-measure of the empty set is zero,

(iii) $\phi(A) \leqq \phi(B)$ whenever $A \subset B \subset E_n$,

(iv) $\phi(\bigcup_{j=1}^{\infty} A_j) \leqq \sum_{j=1}^{\infty} \phi(A_j)$ whenever $A_j \subset E_n$, $j = 1,2,\cdots$.

As usual, a set A is *ϕ-measurable* when $\phi(X) = \phi(X \cap A) + \phi(X - A)$ whenever $X \subset E_n$.

We denote by U_n' the class of all measures ϕ over E_n such that all closed subsets of E_n are ϕ-measurable. By U_n'' we denote the class of all elements $\phi \in U_n'$ with the additional property $\phi E_n < \infty$.

For any set $A \subset E_n$, and $s \geqq 0$, $\mathscr{H}_n^s A$ denotes the Hausdorff s-dimensional measure of A, and this measure is defined in the usual way. Also, $\mathscr{S}_n^s A$

denotes the Hausdorff spherical s-dimensional measure, where the covering sets are restricted to be open spheres. These measures are defined in Federer [2] for integers s, but this restriction is irrevelant in the definitions.

1.5. *Densities.* Given $s \geqq 0, \phi \in U_n', A \subset E_n, x \in E_n$; we have the (ϕ, s) upper and lower spherical densities of A at x defined by

$$\overline{\odot}_n^s(\phi, A, x) = \limsup_{r \to 0} r^{-s} \phi(A \cap C_n(x, r)),$$

$$\underline{\odot}_n^s(\phi, A, x) = \liminf_{r \to 0} r^{-s} \phi(A \cap C_n(x, r)).$$

When these are equal,

$$\odot_n^s(\phi, A, x) = \lim_{r \to 0} r^{-s} \phi(A \cap C_n(x, r)).$$

These densities differ from those defined by Federer by a factor dependent only upon n and s.

We say that x is a (ϕ, s) *regular point with respect to* A when $0 < \odot_n^s(\phi, A, x) < \infty$. In the special case $A = E_n$ we simply call x a (ϕ, s) *regular point.* The set B is (ϕ, s) *regular* if:

(i) B is ϕ-measurable,

(ii) $\phi B < \infty$,

(iii) ϕ-almost all points of B are regular.

Given also an integer k such that $0 \leqq k \leqq n$, we say that x is a *weakly* (ϕ, s, k) *tangential point with respect to* A when it is a (ϕ, s) regular point with respect to A, and, in addition, for some $R \in \mathscr{G}_n$,

$$\liminf_{r \to 0} r^{-s} \phi(A \cap M_n^k(x, R, \eta r, r)) = 0 \text{ whenever } \eta > 0.$$

Again, when $A = E_n$, x is a *weakly* (ϕ, s, k) *tangential point.* The set B is *weakly* (ϕ, s, k) *tangential* if it is (ϕ, s) regular and ϕ-almost all its points are weakly (ϕ, s, k) tangential.

1.6. The expression $y = O(x)$ means that $|y| < K_{n,s} x$, where $K_{n,s}$ depends only on n and s.

2. The purpose of this paper is to prove the following:

THEOREM 1. *Let $\phi \in U_n', s \geqq 0$, and let every point of B be (ϕ, s) regular with respect to A, where $B \subset A \subset E_n$ and $\phi B > 0$. Then*

(i) *s is an integer, and*

(ii) *ϕ-almost all points of B are weakly (ϕ, s, s) tangential with respect to A.*

In [3] I have proved (i) of this theorem in the case $n = 2, \phi = \mathscr{H}_2^s$. The same method would yield a proof for arbitrary ϕ, but would not generalise to n dimensions, nor would it prove (ii). Nevertheless, some of the techniques used in that paper are generalised in the present paper.

We could call a point strongly (ϕ, s, k) tangential if we were able to replace the "lim inf" in the definition by "lim". Then we would have, in the case $s = k$, a definition equivalent ϕ-almost everywhere to Federer's (ϕ, k) restrictedness. The problem of proving (ii) of Theorem 1 with "weakly" replaced by "strongly" still remains open. In [4] I have proved this in the case $n = 3, s = 2, \phi = \mathscr{H}_3^2$, but even then only with a stronger definition of regularity, for I assume that the density actually equals one almost everywhere. Besicovitch [1], Morse and Randolph [7], and Moore [5] have solved the problem completely in the case $s = 1$.

3. THEOREM 2. *Let $\phi \in U_n''$, $s \geqq 0$, and let $B \in \mathscr{B}_n$ be a (ϕ, s) regular set with $\phi B > 0$. Then*
(i) *s is an integer, and*
(ii) *B is weakly (ϕ, s, s) tangential.*

LEMMA A. *The Theorems 1 and 2 are equivalent.*

Proof. We must prove that in Theorem 1 we may assume without loss of generality that

$$\phi A < \infty, \qquad A = E_n \text{ and } B \in \mathscr{B}_n. \tag{1}$$

Accordingly, let us suppose the hypotheses of Theorem 1 are satisfied. Then for every point $a \in B$ we can find r such that

$$\phi[A \cap C_n(a, r)] < \infty,$$

and hence we may cover B by a countable set of such spheres

$$C_n^{(j)}, \quad j = 1, 2, \cdots.$$

Assume Theorem 1 is true in the case $\phi A < \infty$. Then even if $\phi A = \infty$, (i) and (ii) are true with A and B replaced by

$$A \cap C_n^{(j)} \text{ and } B \cap C_n^{(j)}$$

respectively, provided

$$\phi[B \cap C_n^{(j)}] > 0.$$

Summing over j gives us (i) and (ii) as stated, and so we may assume without loss of generality that $\phi A < \infty$.

Again, assume the hypotheses of Theorem 1 are satisfied. Let μ be the measure such that for any set $E \subset E_n$,

$$\mu E = \phi(E \cap A).$$

Let B' be the set of all (μ, s) regular points of E_n. Then $B \subset B' \in \mathscr{B}_n$. (This is easily proved by standard methods.) The hypotheses of Theorem 1 are now also satisfied with ϕ, A, B replaced by μ, E_n, B'.

Consequently, all of (1) may be assumed without loss of generality, and our lemma is proved.

4. Elementary lemmas.

LEMMA 1. *Let $\phi \in U''_n$, $s \geqq 0$, and let every point of $B \in \mathscr{B}_n$ be a (ϕ, s) regular point of E_n. Then for any set $A \subset E_n$ we have*

$$(1) \qquad \odot^s_n(\phi, A - B, x) = 0$$

at ϕ-almost all points $x \in B$.

Proof. From Federerer [2, §3.2], we have (1) holding at $\mathscr{S}^s_n$-almost all x in B. Let the exceptional set be $X \subset B$. Then

$$\mathscr{S}^s_n X = 0 \text{ and hence } \mathscr{H}^s_n X = 0.$$

It remains to prove $\phi X = 0$.

Let X_j denote the points $x \in X$ such that

$$\odot^s_n(\phi, E_n, x) < 1/j.$$

Then

$$X = \bigcup_{j=1}^{\infty} X_j.$$

Further, using Federer [2, §3.6],

$$\phi X_j \leqq 2^{-s} j^{-1} \mathscr{H}^s_n X_j = 0.$$

We deduce $\phi X = 0$ by summing over j.

LEMMA 2. *Let $\phi \in U''_n$ and $B \in \mathscr{B}_n$. Then given $\varepsilon > 0$ we can find a closed set $F \subset B$ such that $\phi F \geqq (1 - \varepsilon)\phi B$.*

For a proof, see [6].

5. **Special notation.** *Given integers $k \leqq n$ and a real number $s \geqq 0$ then $P(n, s, k)$ denotes the following proposition:*

For every measure $\phi \in U''_n$ all (ϕ, s) regular Borel subsets of E_n are weakly (ϕ, s, k) tangential.

6. We devote this section to proving

LEMMA B. *If $0 \leqq s < n$, then $P(n, s, n-1)$ is true.*

LEMMA 3. *Let $\phi \in U'_n$, $a \in E_n$, $r > 0$, and let $f(\rho)$ be a $\mathscr{B}_n$-measurable function of $\rho \in E_1$. Then*

$$(1) \qquad \int_0^r f(\rho)\phi[C_n(a,\rho)]d\rho = \int_{C_n(a,r)} \int_{|x-a|}^r f(\rho)d\rho d\phi x.$$

Proof. Let

$$g(\rho,x) = \begin{cases} f(\rho) & \text{when } |x-a| \leqq \rho, \\ 0 & \text{when } |x-a| > \rho. \end{cases}$$

Then the left-hand side of (1) may be written

$$\int_0^r \int_{C_n(a,r)} g(\rho,x)\, d\phi x\, d\rho.$$

By Fubini's theorem (see Saks [8]), this is equal to

$$\int_{C_n(a,r)} \int_0^r g(\rho,x)\, d\rho d\phi x,$$

which is seen to equal the right-hand side of (1) as required.

Lemma 4. *Let $\phi \in U_n'$, $s \geqq 0$, $l > 0$, $r > 0$, $\varepsilon > 0$, and let $a_0 = 0$ and a_1 be points in E_n such that $|a_1| < r$ and*

$$(1-\varepsilon)l\rho^s < \phi[C_n(a_j,\rho)] < (1+\varepsilon)l\rho^s \text{ whenever } \rho \leqq r,\ j = 0,1.$$

Then

$$\int_{C_n(a_0,r)} x \cdot a_1 d\phi x = O(|a_1|^2 l r^s + \varepsilon l r^{s+2}).$$

Proof. We have for $j = 0,1$,

$$\begin{aligned}\int_{C_n(a_j,r)} (r^2 - |x-a_j|^2)d\phi x &= 2\int_{C_n(a_j,r)} \int_{|x-a_j|}^r \rho d\rho d\phi x \\ &= 2\int_0^r \rho\phi[C_n(a_j,\rho)]d\rho, \text{ by Lemma 3,} \\ &= 2\int_0^r (1+O(\varepsilon))l\rho^{s+1}d\rho = \frac{2l}{s+2}r^{s+2} + O(\varepsilon l r^{s+2}).\end{aligned}$$

Thus

$$(1) \qquad \int_{C_n(a_0,r)} (r^2 - |x|^2)d\phi x - \int_{C_n(a_1,r)} (r^2 - |x-a_1|^2)d\phi x = O(\varepsilon l r^{s+2}).$$

Now for all $x \in C_n(a_0, r+|a_1|) - C_n(a_0, r-|a_1|)$ we have

$$|r^2 - |x-a_1|^2| = |(r+|x-a_1|)(r-|x-a_1|)| \leqq (2r+2|a_1|)(2|a_1|) < 8r|a_1|.$$

Consequently,

$$\begin{aligned}
&\left|\int_{C_n(a_0,r)}(r^2-|x-a_1|^2)d\phi x-\int_{C_n(a_1,r)}(r^2-|x-a_1|^2)d\phi x\right| \\
&\qquad\leqq 8r|a_1|\phi[C_n(a_0,r+|a_1|)-C_n(a_0,r-|a_1|)] \\
&\qquad= 8lr|a_1|[(r+|a_1|)^s-(r-|a_1|)^s+O(\varepsilon r^s)] \qquad (2)\\
&\qquad= O(|a_1|^2lr^s+\varepsilon lr^{s+2}).
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_{C_n(a_0,r)}(r^2-|x-a_1|^2)d\phi x-\int_{C_n(a_0,r)}(r^2-|x|^2)d\phi x \\
&\qquad= \int_{C_n(a_0,r)}(|x|^2-|x-a_1|^2)d\phi x = \int_{C_n(a_0,r)}(2x\cdot a_1-|a_1|^2)d\phi x \\
&\qquad= -|a_1|^2\phi[A\cap C_n(a_0,r)]+2\int_{C_n(a_0,r)}x\cdot ad\phi x \\
&\qquad= -|a_1|^2lr^s+O(\varepsilon lr^{s+2})+2\int_{C_n(a_0,r)}x\cdot ad\phi x.
\end{aligned}$$

We can now deduce the lemma by applying (1) and (2).

LEMMA 5. *Let $\phi\in U_n'$, $s\geqq 0$, $l>0$, $r>0$, $\varepsilon>0$ and let $a_0=0,a_1,\cdots,a_n$ be points in E_n such that*

$$\alpha = \max_{j=1,\dots,n}|a_j|<r,$$

$$\Delta = \Delta(a_1,\cdots,a_n)\neq 0,$$

$(1-\varepsilon)l\rho^s<\phi[C_n(a_j,\rho)]<(1+\varepsilon)l\rho^s$ *whenever* $\rho\leqq r, j=0,1,\cdots,n$.
Then for any direction θ,

$$\int_{C_n(a_0,r)}x\cdot\theta d\phi x=O(\alpha^{n+1}|\Delta|^{-1}lr^s+\varepsilon\alpha^{n-1}|\Delta|^{-1}lr^{s+2}).$$

Proof. As usual, we let $x=(x^{(1)},\cdots,x^{(n)})$ and $a_j=(a_j^{(1)},\cdots,a_j^{(n)})$. Then

$$\sum_{\lambda=1}^n a_j^{(\lambda)}\int_{C_n(a_0,r)}x^{(\lambda)}d\phi x=\int_{C_n(a_0,r)}\sum_{\lambda=1}^n x^{(\lambda)}a_j^{(\lambda)}d\phi x = \int_{C_n(a_0,r)}x\cdot a_jd\phi x$$

$$= O(\alpha^2lr^s+\varepsilon lr^{s+2}) \text{ for } j=1,\cdots,n,$$

by Lemma 4.

To prove this lemma for any given direction θ we may assume without loss of generality that axes have been set up so that $\theta=(1,0,\cdots,0)$. Then regarding $a_j^{(\lambda)}$ as coefficients of linear equations, we have

$$\int_{C_n(a_0,r)} x \cdot \theta d\phi x = \int_{C_n(a_0,r)} x^{(1)} d\phi x$$
$$= O[(\alpha^2 l r^s + \varepsilon l r^{s+2})\alpha^{n-1}|\Delta|^{-1}],$$

as required.

In the following lemma we modify slightly the definition given in 1.5 of weakly tangential points, and show that the new definition is equivalent.

LEMMA 6. *Let $\phi \in U_n'$, $x \in E_n$, $s \geqq 0$ and let k be an integer such that $0 \leqq k \leqq n$. Then x is a weakly (ϕ, s, k) tangential point if and only if*

(i) *x is (ϕ, s) regular, and*

(ii) *for some function $R' = R'(\eta, r) \in \mathscr{G}_n$,*

(1) $$\liminf_{r \to 0} r^{-s}\phi[M_n^k(x, R', \eta r, r)] = 0 \text{ whenever } \eta > 0.$$

The only difference is that in the original definition R was independent of η and r.

Proof. Suppose that our new definition is satisfied.

Let $R_j = R'(1/j,\ 1/j)$ for $j = 1, 2, \cdots$, and let R be any limit point in $\mathscr{G}_n$ of this sequence. Then it is easily seen that (1) holds with R' replaced by R, and so our original definition in 1.5 is satisfied.

The implication in the other direction is trivial.

LEMMA 7. *Let $a_0 = 0 \in E_n$, $H \subset E_n$, $\lambda > 0$ be such that for all directions θ,*

$$H \cap M_n(a_0, \theta, \lambda) \text{ is not empty.}$$

Then we can find points $a_1, \cdots, a_n \in H$ such that

$$|\Delta(a_1, \cdots, a_n)| > \lambda^n.$$

Proof. Let θ_1 be an arbitrary direction. Then we can find a point

$$a_1 \in H \cap M_n(a_0, \theta_1, \lambda),$$

which implies $|a_1 \cdot \theta_1| > \lambda$.

Let θ_2 be any direction such that

$$a_1 \cdot \theta_2 = 0.$$

Then we can find a point

$$a_2 \in H \cap M_n(a_0, \theta_2, \lambda)$$

which implies

$$|a_2 \cdot \theta_2| > \lambda.$$

In general, having found a_j, θ_j for $j = 1, \cdots, p < n$ such that $|a_j \cdot \theta_j| > \lambda$ for $j = 1, \cdots, p$ and $a_j \cdot \theta_j = 0$ for $i < j = 1, \cdots, p$, we let θ_{p+1} be any direction such that $a_i \cdot \theta_{p+1} = 0$ for $i = 1, \cdots, p$. Then we can find a point

$$a_{p+1} \in H \cap M_n(a_0, \theta_{p+1}, \lambda),$$

which implies

$$|a_{p+1} \cdot \theta_{p+1}| > \lambda.$$

In this way we find $a_j \in H$ and θ_j, for $j = 1, \cdots, n$, such that

$$|a_j \cdot \theta_j| > \lambda \text{ for } j = 1, \cdots, n$$

and $a_i \cdot \theta_j = 0$ for $i < j = 1, \cdots, n$. That is,

$$\begin{bmatrix} a_1^{(1)} \cdots a_1^{(n)} \\ \vdots \quad\quad \vdots \\ a_n^{(1)} \cdots a_n^{(n)} \end{bmatrix} \cdot \begin{bmatrix} \theta_1^{(1)} \cdots \theta_n^{(1)} \\ \vdots \quad\quad \vdots \\ \theta_1^{(n)} \cdots \theta_n^{(n)} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix},$$

where $|\alpha_{jj}| > \lambda$ and $\alpha_{ij} = 0$ whenever $i < j$.

Taking determinants,

$$|\Delta(a_1, \cdots, a_n)\Delta(\theta_1, \cdots, \theta_n)| = |\alpha_{11} \cdots \alpha_{nn}| > \lambda^n.$$

On the other hand, by a well-known theorem on determinants,

$$|\Delta(\theta_1, \cdots, \theta_n)|^2 \leqq \prod_{j=1}^{n} \sum_{i=1}^{n} (\theta_j^{(i)})^2 = 1,$$

since the θ_j are directions.

Our lemma now follows.

LEMMA 8. *Let $\phi \in U_n''$, $s \geqq 0$, and let A be the set of those points in E_n which are (ϕ, s) regular but nonweakly $(\phi, s, n-1)$ tangential. Then at ϕ-almost every point $a \in A$ we have*

$$\lim_{r \to 0} r^{-(s+1)} \int_{C_n(a,r)} (x - a) \cdot \theta d\phi x = 0 \textit{ for all directions } \theta.$$

Proof. Suppose the lemma is false. Then we can find a set $A^* \subset A$, $A^* \in \mathscr{B}_n$, $\lambda > 0$, $\mu > 0$ such that $\phi A^* > 0$, and at every point $a \in A^*$ we have

$$(1) \qquad \odot_n^s(\phi, E_n, a) < \lambda$$

and

$$(2) \qquad \limsup_{r \to 0} r^{-(s+1)} \int_{C_n(a,r)} (x - a) \cdot \theta d\phi x > \mu$$

for some θ.

By Lemmas 2 and 6 we find a closed set $B \subset A^*$, and η, where $0 < \eta < 1$, such that

$$\phi B > 0$$

and

(3) $$\liminf_{r \to 0} r^{-s} \phi[M_n(a, \theta, \eta\rho, \rho)] > 0 \text{ for any } \theta \text{ and whenever } a \in B.$$

Now let ε be arbitrary subject to $0 < \varepsilon < 1$. Since (1) holds at every point $a \in B$ we can find a closed set $D \subset B$, $\delta > 0$, and l such that

(4) $$0 < l < \lambda, \qquad \phi D > 0,$$

and

(5) $$(1 - \varepsilon) l \rho^s < \phi C_n(a, \rho) < (1 + \varepsilon) l \rho^s \text{ whenever } a \in D \text{ and } \rho < \delta.$$

Let a_0 be any point of D at which (use Lemma 1)

$$\odot_n^s(\phi, E_n - D, a_0) = 0.$$

Since (3) holds at a_0 we now have

$$\liminf_{r \to 0} r^{-s} \phi[D \cap M_n(a, \theta, \eta\rho, \rho)] > 0$$

for all θ, and hence for some δ_1, where $0 < \delta_1 < \delta$,

(6) $$\phi[D \cap M_n(a, \theta, \eta\rho, \rho)] > 0 \text{ whenever } p < \delta_1 \text{ and for all } \theta.$$

Take now any $r < \delta_1$ and let

(7) $$\rho_1 = \varepsilon^{1/2} r, \qquad H = D \cap C_n(a_0, \rho_1).$$

Take axes so that $a_0 = 0$. Apply (6) with $\rho = \rho_1$. Then for all θ,

$$H \cap M_n(a_0, \theta, \eta\rho_1) \text{ is not empty.}$$

Consequently, by Lemma 7 we can find points $a_1, \cdots, a_n \in D$ such that

$$\alpha = \max_{j = 1, \dots, n} |a_j| < \rho_1 < r$$

and

$$|\Delta| = |\Delta(a_1, \cdots, a_n)| > \eta^n \rho_1^n.$$

In addition, (5) holds at each a_j, and so by Lemma 5, for any direction θ,

$$\begin{aligned}\int_{C_n(a_0,r)} x\cdot\theta d\phi x &= O(\rho_1^{n+1}\eta^{-n}\rho_1^{-n}lr^s+\varepsilon\rho_1^{n-1}\eta^{-n}\rho_1^{-n}lr^{s+2})\\ &= O(\rho_1\eta^{-n}lr^s+\varepsilon\rho_1^{-1}\eta^{-n}lr^{s+2})\\ &= O(\varepsilon^{1/2}\eta^{-n}\lambda r^{s+1}) \text{ from (4) and (7).}\end{aligned}$$

Since this holds for any $r<\delta_1$ we have

$$\limsup_{r\to 0} r^{-(s+1)}\int_{C_n(a_0,r)} x\cdot\theta d\phi x = O(\varepsilon^{1/2}\eta^{-n}\lambda) \text{ for all } \theta.$$

Hence from (2), $\mu=O(\varepsilon^{1/2}\eta^{-n}\lambda)$. But ε was chosen arbitrarily small after λ, μ and η had been determined. Thus we have a contradiction and the lemma must be true.

Lemma 9. *If $\phi\in U_n''$, $0\leqq s<n$, then at ϕ-almost every (ϕ,s) regular point $a\in E_n$ we can find a direction θ (depending on a) such that*

$$\liminf_{r\to 0} r^{-s}\phi[H_n(a,\theta,\eta r,r)] = 0 \textit{ whenever } \eta>0.$$

Proof. Let l,λ,η,r_1 be arbitrary subject to $l,\lambda,r_1>0$, $0<\eta<1$, and let A_1 denote the set of points $a\in E_n$ such that

$$\odot_n^s(\phi,E_n,a)<l$$

and

$$\phi[H_n(a,\theta,\eta r,r)]>\lambda r^s \text{ for all } \theta \text{ whenever } r<r_1. \tag{1}$$

Suppose, contrary to the lemma, that $\phi A_1>0$, and choose a closed set $B_1\subset A_1$, and r_2 such that

$$\phi B_1>0, \qquad 0<r_2<r_1,$$

and

$$\phi[C_n(a,r)]<2lr^s \text{ whenever } a\in B_1 \text{ and } r<r_2. \tag{2}$$

Let a_1 be any point of B_1 at which

$$\odot_n^s(\phi,E_n-B_1,a_1)=0.$$

Then we can, given $\varepsilon>0$, find r_3 such that

$$0<r_3<r_2,$$

and

$$\phi[(E_n-B_1)\cap C_n(a_1,r_3)]<\varepsilon r_3^s. \tag{3}$$

Now let ρ denote the greatest distance of any point in $C_n(a_1,(1/3n)r_3)$ from B_1.

We shall determine a lower bound for ρ. Note on the other hand that $\rho\leqq(1/3n)r_3$.

Take a_1 at 0 and consider all points $(6m_1\rho,\cdots,6m_n\rho)$ in $C_n(a_1,(1/3n)r_3)$, where $m_1,\cdots,m_n$ are integers. The number of these points exceeds $K\rho^{-n}r_3^n$, where K depends only on n. Denote the points by p_j, $j=1,\cdots,t$. By the definition of ρ, there is a point of B_1 within ρ of every

$$p_j = (6m_1\rho,\cdots,6m_n\rho).$$

From (1) it follows that the cube bounded by $x_i=(6m_i\pm 2)\rho$, $i=1,\cdots,n$, has ϕ-measure exceeding $\lambda\rho^s$. Summing over all the cubes, which are nonoverlapping, and all contained in $C_n(a_1,r_3)$,

$$\phi[C_n(a_1,r_3)] > (K_\rho^{-n}r_3^n)(\lambda\rho^s) = K\lambda\rho^{s-n}\ r_3^n.$$

On the other hand, from (2) we have

$$\phi[C_n(a_1,r_3)] < 2lr_3^s,$$

and hence

$$\rho > \left(\frac{K\lambda}{2l}\right)^{1/(n-s)} r_3.$$

It follows that we can find a point b_1 in $C_n(a_1,(1/3n)r_3)$ whose distance, ρ_1 say, from B_1 satisfies

$$\left(\frac{K\lambda}{2l}\right)^{1/(n-s)} r_3 < \rho_1 < \left(\frac{1}{3n}\right) r_3. \tag{4}$$

Let b_2 be the point of B_1 (or one of them) which lies on the boundary of $C_n(b_1,\rho_1)$. Since the interior of this sphere contains no points of B_1 it follows by geometry, with θ_1 the direction of b_2b_1, that

$$B_1 \cap H_n(b_2,\theta_1,\eta^2\rho_1,\eta\rho_1) \text{ is empty.}$$

Also, this set is contained in $C_n(a_1,r_3)$, and hence from (3) we have

$$\begin{aligned}\phi[H_n(b_2,\theta_1,\eta^2\rho_1,\eta\rho_1)] &< \varepsilon r_3^s \\ &< \varepsilon\left(\frac{2l}{K\lambda}\right)^{s/(n-s)} \eta^{-s}(\eta\rho_1)^s, \text{ from (4).}\end{aligned}$$

Now ε was chosen independently of l, λ, η, and if chosen sufficiently small, then the above inequality will contradict (1) when $\theta=\theta_1$ and $r=\eta\rho_1<r_1$. Hence our assumption $\phi A_1>0$ must be false. We now refer to the definition of A_1. Since the (ϕ,s) density is finite at every (ϕ,s) regular point, and l may be arbitrarily large, we have, for arbitrary λ,η,r_1 and ϕ-almost every (ϕ,s) regular point a,

$$\phi[H_n(a,\theta,\eta r,r)] \leqq \lambda r^s \text{ for some } \theta \text{ and some } r<r_1.$$

For every positive integer m, let $\lambda^{(m)} = r_1^{(m)} = m^{-1}$. Then for ϕ-almost every (ϕ,s) regular point a we can find a direction $\theta^{(m)}$ and a positive number $r^{(m)} < r_1^{(m)}$ such that

$$\phi[H_n(a,\theta^{(m)},\eta r^{(m)},r^{(m)})] \leqq \lambda^{(m)}(r^{(m)})^s.$$

Let θ denote any one of the limiting directions of the sequence $\{\theta^{(m)}\}$. Then

$$\liminf_{r\to 0} r^{-s}\phi[H_n(a,\theta,2\eta r,r)] = 0,$$

and since η can be arbitrarily small the lemma is proved.

We can now prove Lemma B which is stated at the beginning of this section.

Proof of Lemma B. Let $\phi \in U_n''$, $0 \leqq s < n$, and let A be the set of those points which are (ϕ,s) regular but not weakly $(\phi,s,n-1)$ tangential. We must prove that $\phi A = 0$.

By Lemma 8, at ϕ-almost every point $a \in A$ we have

$$\lim_{r\to 0} r^{-(s+1)} \int_{C_n(a,r)} (x-a)\cdot\theta d\phi x = 0$$

for all direction θ.

Also, by Lemma 9, at ϕ-almost every point $a \in A$ we can find a direction θ (depending on a) such that

$$\liminf_{r\to 0} r^{-s}\phi[H_n(a,\theta,\eta r,r)] = 0 \text{ whenever } \eta > 0.$$

By regularity, at every point $a \in A$ we can find a positive number l (depending on a) such that

$$\lim_{r\to 0} r^{-s}\phi[C_n(a,r)] = l.$$

Thus, at ϕ-almost every point $a \in A$, we can, given ε,η, where $\varepsilon > 0$ and $0 < \eta < 1$, find arbitrarily small $r > 0$ such that

$$\left| \int_{C_n(a,r)} (x-a)\cdot\theta d\phi x \right| < \varepsilon r^{s+1}, \tag{1}$$

$$\phi[H_n(a,\theta,\eta r,r)] < \varepsilon r^s, \tag{2}$$

and

$$\phi[C_n(a,r)] < 2lr^s. \tag{3}$$

Let us use the notation $C = C_n(a,r)$, $H = H_n(a,\theta,\eta r,r)$ and $H^* = H_n(a,-\theta,\eta^{1/2}r,r)$.

Then

$$\int_C (x-a)\cdot\theta d\phi x = \int_H (x-a)\cdot\theta d\phi x + \int_{H^*} (x-a)\cdot\theta d\phi x$$
$$+ \int_{C-H-H^*} (x-a)\cdot\theta d\phi x$$
$$\leqq r\phi H - \eta^{1/2} r\phi H^* + \eta r\phi(C-H-H^*).$$

Hence, from (1), (2) and (3) (which hold for some arbitrarily small r),

$$-\varepsilon r^{s+1} < \varepsilon r^{s+1} - \eta^{1/2} r\phi H^* + 2l\eta r^{s+1},$$

hence

$$\phi H^* < 2(\varepsilon\eta^{-1/2} + l\eta^{1/2})r^s.$$

We could have chosen $\varepsilon = \eta$, and then

$$\phi H^* = \phi[H_n(a, -\theta, \eta^{1/2}r, r)] < 2(l+1)\eta^{1/2} r^s.$$

Adding this to (2) we have

$$\phi[M_n(a,\theta,\eta^{1/2}r,r)] < [2(l+1)\eta^{1/2} + \eta]r^s,$$

which holds at ϕ-almost all points $a \in A$ for a direction θ and a set of arbitrarily small r. Since η was chosen arbitrarily, and the left-hand side increases as $\eta \to 0$, it follows that

$$\liminf_{r\to 0} r^{-s}\phi[M_n(a,\theta,\lambda r,r)] = 0 \text{ whenever } \lambda > 0.$$

With regularity, this implies that almost all points of A are weakly $(\phi, s, n-1)$ tangential. By the definition of A, we now have $\phi A = 0$, and the proof is complete.

7. In this section we generalise Lemma B by proving

LEMMA C. *If s is a number and k an integer such that $0 \leqq s < k+1 \leqq n$, then $P(n,s,k)$ is true.*

LEMMA 10. *Given a sequence of measures $\phi_j \in U_n''$, $j = 1,2,\cdots$, such that*

$$\lim_{j\to\infty} \phi_j[C_n(0,1)] = 1,$$

and

$$\phi_j A = 0 \textit{ whenever } A \cap C_n(0,1) \textit{ is empty},$$

we can find a subsequence of integers, j_m, $m = 1,2,\cdots$, and a measure $\phi \in U_n''$ such that

$$\limsup_{m\to\infty} \phi_{j_m} A \leqq \phi B \textit{ and } \phi A \leqq \liminf_{m\to\infty} \phi_{j_m} B \textit{ whenever } \mathrm{Cl}(A) \subset \mathrm{Int}(B).$$

Proof. By a well-known theorem on integration theory we can choose a subsequence $\phi_{j_m} = \mu_m$, say, $m = 1, 2, \cdots$, and a measure $\phi \in U_n''$ so that for any continuous function $f(x)$ with domain E_n,

$$\lim_{m \to \infty} \int f(x) d\mu_m = \int f(x) d\phi. \tag{1}$$

Let $\mathrm{Cl}(A) \subset \mathrm{Int}(B)$. Then we can define $f(x)$ as follows:

$$f(x) = \begin{cases} 1 & \text{whenever } x \in \mathrm{Cl}(A), \\ 0 & \text{whenever } x \in E_n - \mathrm{Int}(B). \end{cases}$$

We can extend $f(x)$ to the whole of E_n, so that $f(x)$ is continuous and $0 \leqq f(x) \leqq 1$. Then

$$\mu_m A \leqq \int f(x) d\mu_m,$$

and hence, by (1),

$$\limsup_{m \to \infty} \mu_m A \leqq \int f(x) d\phi \leqq \phi B,$$

as required, and the second part is proved similarly.

LEMMA 11. *If s is a number and k an integer such that $0 \leqq s < k \leqq n$, then $P(n,s,k)$ implies $P(n,s,k-1)$.*

Proof. Let us suppose the lemma is not true. Then there exist s, k such that $0 \leqq s < k \leqq n$, $P(n,s,k)$ is true but $P(n,s,k-1)$ is false.

Under this hypothesis we will construct a measure $\phi \in U_k''$, and show that $P(k,s,k-1)$ is false. This will contradict Lemma B with $n = k$.

First, however, we construct a sequence of measures ϕ_j, of which a subsequence will converge to ϕ. We do this in the following

ASSERTION. *We can find $\eta_0, K_0, > 0$, a sequence of measures $\phi_j \in U_k''$, and of closed sets $D_j \subset E_k$, $j = 1, 2, \cdots$, such that*

$$0 \in D_j, \quad \lim_{j \to \infty} \phi_j[D_j \cap C_k(q_j, r)] = r^s, \quad \lim_{j \to \infty} \phi_j[D_j \cap C_k(0,1)] = 1, \tag{1}$$

and

$$\liminf_{j \to \infty} \phi_j[D_j \cap M_k^{k-1}(q_j, S, \eta_0 r, r)] > K_0 r^s, \tag{2}$$

whenever $q_j \in D_j \cap C_k\ (0, 1/2)$, $0 < r \leqq 1/2$ and $S \in \mathscr{G}_k$.

We now prove this Assertion.

Since $P(n,s,k-1)$ is false, for some $\phi \in U_n''$ we can find a (ϕ, s) regular set $A^* \in \mathscr{B}_n, \phi A^* > 0$, none of whose points are weakly $(\phi, s, k-1)$ tangential. On the other hand, by $P(n,s,k)$ we have that A^* is weakly (ϕ, s, k) tangential.

At every point $a \in A^*$, we can find $K, \eta_0 > 0$ such that

$$\liminf_{r \to 0} r^{-s} \phi[A \cap M_n^{k-1}(a, R, \eta_0 r, r)] > K \text{ whenever } R \in \mathscr{G}_n.$$

By taking a suitable subset $A \subset A^*$, with $\phi A > 0$, we may assume that the above holds uniformly at every $a \in A$ for some K, η_0 independent of a. Similarly, we may assume that the (ϕ, s) density, which is always positive by regularity, is always less than some $l > 0$ at every $a \in A$.

For each positive integer j, we can now find a closed set $A_j \subset A$, $r_j > 0$ and l_j such that

$$0 < l_j < l, \quad \phi A_j > 0, \tag{3}$$

and at every point $a \in A_j$,

$$|\phi[C_n(a, r)] - l_j r^s| < l_j(r^s/j) \text{ whenever } r < r_j, \tag{4}$$

$$\phi[M_n^{k-1}(a, R, \eta_0 r, r)] > K r^s \text{ whenever } R \in \mathscr{G}_n \text{ and } r < r_j, \tag{5}$$

and

$$\liminf_{r \to 0} r^{-s} \phi[M_n^k(a, R, (1/4j) r, r)] < l_j/j \text{ for some } R \in \mathscr{G}_n. \tag{6}$$

For each j, let a_j be a point of A_j at which

$$\odot_n^s(\phi, E_n - A_j, a_j) = 0.$$

Then we can find $\rho_j < \frac{1}{2} r_j$ such that

$$\phi[(E_n - A_j) \cap C_n(a_j, 4\rho_j)] < l_j(\rho_j^s/j), \tag{7}$$

and (using (6)) such that for some $R_j \in \mathscr{G}_n$,

$$\phi[M_n^k(a_j, R_j, (1/j)\rho_j, 4\rho_j)] < 4^s l_j(\rho_j^s/j),$$

and with (7) this gives

$$\phi[A_j \cap M_n^k(a_j, R_j, (1/j)\rho_j, 4\rho_j)] = O(l_j(\rho_j^s/j)). \tag{8}$$

Let a be any point in $A_j \cap C_n(a_j, 2\rho_j)$, and ρ any number such that

$$0 < \rho \leqq 2\rho_j.$$

Then

$$C_n(a, \rho) \subset C_n(a_j, 4\rho_j),$$

and so from (4) and (7),

$$|\phi[A_j \cap C_n(a, \rho)] - l_j \rho^s| < (l_j/j)(\rho^s + \rho_j^s) = O(l_j(\rho_j^s/j)),$$

hence

$$\phi[A_j \cap C_n(a,\rho)] = l_j[\rho^s + O(\rho_j^s/j)]. \tag{9}$$

Also, from (5) and (7),

$$\phi[A_j \cap M_n^{k-1}(a,R,\eta_0\rho,\rho)] > K\rho^s - l_j(\rho_j^s/j) \text{ whenever } R \in \mathscr{G}_n. \tag{10}$$

For each j we next define a special measure $\mu_j \in U_k''$, and a set $B_j \subset E_k$.
To simplify, we assume that axes are such that $a_j = 0$ and $R_j = I_n$. Also, let

$$L_n^k(a_j,R_j,(1/j)\rho_j,4\rho_j) = L.$$

For any set $B \subset E_k$, we define

$$\mu_j B = \phi[(B \times E_{n-k}) \cap A_j \cap L].$$

Let $B_j = P_n^k(A_j \cap L)$. Assume always $j > n$. Then for any $b \in B_j \cap C_k(0,\rho_j)$ we can find

$$a \in A_j \cap L \cap C_n(a_j,2\rho_j)$$

such that $a \in \{b\} \times E_{n-k}$.

For any positive $\lambda < \rho_j$ we have

$$C_n(a,\lambda) \cap L \subset (C_k(b,\lambda) \times E_{n-k}) \cap L \subset C_n(a,\lambda + (n/j)\rho_j). \tag{11}$$

Recalling that (9) holds for any $a \in A_j \cap C_n(a_j,2\rho_j)$ and positive $\rho < 2\rho_j$, it follows that

$$\begin{aligned}
\mu_j[B_j \cap C_k(b,\lambda)] &= \phi[(C_k(b,\lambda) \times E_{n-k}) \cap A_j \cap L] \\
&\leqq \phi[A_j \cap C_n(a,\lambda + (n/j)\rho_j)], \text{ from (11)}, \\
&= l_j[(\lambda + (n/j)\rho_j)^s + O(\rho_j^s/j)], \text{ from (9)}, \\
&= l_j\left[\lambda^s + O\left(\lambda^{s-1}\frac{\rho_j}{j} + \frac{\rho_j^s}{j^s} + \frac{\rho_j^s}{j}\right)\right].
\end{aligned}$$

The error term is rather awkward because $s - 1$ may be negative. But an important property of this term is that it tends to zero as j tends to infinity.

The opposite inequality is

$$\begin{aligned}
\mu_j[B_j \cap C_k(b,\lambda)] &\geqq \phi[A_j \cap L \cap C_n(a,\lambda)] \\
&\geqq \phi[A_j \cap C_n(a,\lambda)] - \phi[A_j \cap M_n^k(a_j,R_j,(1/j)\rho_j,4\rho_j)] \\
&> l_j[\lambda^s + O(\rho_j^s/j)] - O(l_j(\rho_j^s/j)), \text{ from (8) and (9)}, \\
&= l_j[\lambda^s + O(\rho_j^s/j)].
\end{aligned}$$

Consequently,

$$\mu_j[B_j \cap C_k(b,\lambda)] = l_j\left[\lambda^s + O\left(\lambda^{s-1}\frac{\rho_j}{j} + \frac{\rho_j^s}{j^s} + \frac{\rho_j^s}{j}\right)\right] \tag{12}$$

whenever

$$b \in B_j \cap C_k(0,\rho_j),\ \lambda < \rho_j \text{ and } j > n.$$

Note that this is, in a certain sense, an analogue in E_k of (9).

We next obtain a similar analogue of (10). Take then any $S \in \mathcal{G}_k$, and form the matrix

$$R = \begin{bmatrix} S & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{G}.$$

For any $b \in B_j \cap C_k(0,\rho_j)$ we can, as before, find

$$a \in A_j \cap L \cap C_n(a_j, 2\rho_j)$$

such that

$$a \in \{b\} \times E_{n-k}.$$

Next we take, if possible, any λ such that

$$\lambda < \rho_j \text{ and } \lambda > 2\,\frac{\rho_j}{\eta_0 j}. \tag{13}$$

We can now prove that

$$[M_k^{k-1}(b,S,\eta_0\lambda) \times E_{n-k}] \cap L = M_n^{k-1}(a,R,\eta_0\lambda) \cap L. \tag{14}$$

For the set $M_n^{k-1}(a,R,\eta_0\lambda)$ is the set of points x such that, if

$$R(x-a) = y,$$

then $|y^m| > \eta_0\lambda$ for at least one value of $m = k,\cdots,n$. Let the last row of S, the only one which matters, be given by the direction $\theta = (\theta^{(1)},\cdots,\theta^{(k)})$. Then the condition on x is that either

(i) $|\theta_1(x^{(1)} - a^{(1)}) + \cdots + \theta_k(x^{(k)} - a^{(k)})| > \eta_0\lambda$, or

(ii) $|x^{(m)} - a^{(m)}| > \eta_0\lambda$ for at least one value of $m = k+1,\cdots,n$.

Since $a \in L$, we have

$$|a^{(m)}| \leqq (1/j)\rho_j, \quad m = k+1,\cdots,n.$$

Also, from (13) we have $\eta_0\lambda > (2/j)\rho_j$.

Consequently, if possibility (ii) is realised, we have

$$|x^{(m)}| > (1/j)\rho_j$$

for at least one value of $m = k+1,\cdots,n$, which implies $x \notin L$. Thus the set on the right-hand side of (14) is the set of x such that (i) holds, and, in addition,

$$|x^{(m)}| \leqq (1/j)\rho_j, \quad m = k+1, \cdots, n.$$

This is seen to be the same as the set on the left-hand side of (14), and so (14) is proved.

Now,

$$\begin{aligned}
&[M_k^{k-1}(b,S,\eta_0\lambda,\lambda) \times E_{n-k}] \cap L \\
&\qquad = [M_k^{k-1}(b,S,\eta_0\lambda) \times E_{n-k}] \cap [C_k(b,\lambda) \times E_{n-k}] \cap L \\
&\qquad \supset M_n^{k-1}(a,R,\eta_0\lambda) \cap C_n(a,\lambda) \cap L, \text{ from (11) and (14),} \\
&\qquad = M_n^{k-1}(a,R,\eta_0\lambda,\lambda) \cap L.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mu_j[B_j \cap M_k^{k-1}(b,S,\eta_0\lambda,\lambda)] \\
&\qquad = \phi[A_j \cap (M_k^{k-1}(b,S,\eta_0\lambda,\lambda) \times E_{n-k}) \cap L] \\
&\qquad \geqq \phi[A_j \cap M_n^{k-1}(a,R,\eta_0\lambda,\lambda) \cap L] \\
&\qquad \geqq \phi[A_j \cap M_n^{k-1}(a,R,\eta_0\lambda,\lambda)] - \phi[A_j \cap M_n^k(a_j,R_j,(1/j)\rho_j,4\rho_j)] \\
&\qquad > K\lambda^s + O\left(l_j \frac{\rho_j^s}{j}\right), \text{ from (8) and (10).}
\end{aligned}$$

This is true for any λ given by (13). That is, we have

$$\mu_j[B_j \cap M_k^{k-1}(b,S,\eta_0\lambda,\lambda)] > k\lambda^s + O\left(l_j \frac{\rho_j^s}{j}\right) \tag{15}$$

whenever

$$b \in B_j \cap C_k(0,\rho_j) \text{ and } 2\,\frac{\rho_j}{\eta_0 j} < \lambda < \rho_j.$$

Finally, we transform (12) and (15) into (1) and (2), respectively, as follows: Let

$$D_j = \rho_j^{-1} B_j,$$

and let $\phi_j \in U_k''$ be the measure such that for any set A,

$$\phi_j A = \frac{1}{l_j \rho_j^s} \mu_j(\rho_j A).$$

Then for any $q_j \in D_j \cap C_k(0,1)$ we have

$$b = \rho_j q_j \in B_j \cap C_k(0,\rho_j),$$

and for any positive $r < 1$, we have

$$\lambda = \rho_j r < \rho_j.$$

Hence, from (12), provided $j > n$,

$$\mu_j[B_j \cap C_k(\rho_j q_j, \rho_j r)] = l_j\left[\rho_j^s r_j^s + O\left(\rho_j^s\left(\frac{r^{s-1}}{j} + \frac{1}{j^s} + \frac{1}{j}\right)\right)\right].$$

That is,

$$\phi_j[D_j \cap C_k(q_j, r)] = r^s + O\left(\frac{r^{s-1}}{j} + \frac{1}{j^s} + \frac{1}{j}\right),$$

which gives us (1), as required.

Similarly, we deduce from (15), that provided

$$2\frac{\rho_j}{\eta_0 j} < \rho_j r, \text{ or } j > 2\frac{1}{\eta_0 r},$$

then

$$\mu_j[B_j \cap M_k^{k-1}(\rho_j q_j, S, \eta_0\rho_j r, \rho_j r)] > K\rho_j^s r^s + O\left(l_j\frac{\rho_j^s}{j}\right).$$

That is,

$$\phi_j[D_j \cap M_k^{k-1}(q_j, S, \eta_0 r, r)] > \frac{K}{l_j} r^s + O\left(\frac{1}{j}\right).$$

From (3),

$$\frac{K}{l_j} > \frac{K}{l} = K_0, \text{ say,}$$

and (2) now follows.

We have already noted that $\phi \in U_k''$. Since each A_j was closed, the transformed sets D_j are also closed. Finally, $0 \in B_j$ and hence $0 \in D_j$. Since we have established (1) and (2), the proof of our Assertion is complete.

The next step is to deduce from the Assertion that $P(k,s,k-1)$ is false.

For each j, let

$$f_j(x) = \rho(x, D_j \cap C_k(0,1)), \quad x \in E_k.$$

These functions are equicontinuous and hence we can find a convergent subsequence such that

$$\lim_{m\to\infty} f_{j_m}(x) = f(x), \text{ which is continuous.}$$

But to simplify notation we can assume without loss of generality that

$$\lim_{j\to\infty} f_j(x) = f(x),$$

and the Assertion will still hold.

The sets ϕ_j-measured in the Assertion are contained in $D_j \cap C_k(0,1)$. We may therefore assume without loss of generality that

(16) $$\phi_j A = 0 \text{ whenever } A \cap D_j \cap C_k(0,1) \text{ is empty.}$$

Thus we may apply Lemma 10, taking a further subsequence. Or, we may assume without loss of generality that for some measure $\phi \in U_k''$,

(17) $$\limsup_{j\to\infty} \phi_j A \leqq \phi B \text{ and } \phi A \leqq \liminf_{j\to\infty} \phi_j B$$

whenever

$$\mathrm{Cl}(A) \subset \mathrm{Int}(B).$$

Now let D be the set of x at which $f(x) = 0$. (This is a kind of limit of D_j). Take any point $q \in D \cap C_k(0, 1/2)$. At q we shall obtain formulas analogous to (1) and (2) of the Assertion, with ϕ_j replaced by ϕ.

Since $f(q) = 0$, we have

$$\lim_{j\to\infty} \rho(q, D_j) = 0.$$

given $\varepsilon > 0$ we can find $m > 0$ such that

$$\rho(q, D_j) < \varepsilon \text{ whenever } j > m.$$

That is, for each $j > m$ there exists $q_j \in D_j \cap C_k(0, 1/2)$ such that $\rho(q, q_j) < \varepsilon$.

Take any positive $r \leqq 1/2$, so that (1) holds. Also,

$$C_k(q_j, r) \subset C_k(q, r+\varepsilon) \subset K_k(q, r+2\varepsilon).$$

Hence

$$\begin{aligned}\phi[C_k(q, r+2\varepsilon)] &\geqq \phi[K_k(q, r+2\varepsilon)] \\ &\geqq \limsup_{j\to\infty} \phi_j[C_k(q, r+\varepsilon)], \text{ from (17)}, \\ &\geqq \limsup_{j\to\infty} \phi_j[C_k(q_j, r)] \\ &= \limsup_{j\to\infty} \phi_j[D_j \cap C_k(q_j, r)], \text{ from (16)}, \\ &= r^s, \text{ from (1)}.\end{aligned}$$

Since ε is arbitrary, $\phi[C_k(q,r)] \geqq r^s$. Similarly, we can obtain from (1), (16) and (17) the reverse inequality, and so

(18) $$\phi[C_k(q,r)] = r^s.$$

Applying the same technique to (2) and using the fact that for all sufficiently large j, and any $S \in \mathscr{G}_k$,

$$M_k^{k-1}(q, S, \eta_0 r/2, r+\varepsilon) \supset M_k^{k-1}(q_j, S, \eta_0 r, r),$$

we also have

$$\phi[M_k^{k-1}(q,S,\eta_0 r/2,r)] > K_0 r^s.$$

Thus every point of $D \cap C_k(0,1/2)$ is (ϕ,s) regular, but not weakly $(\phi,s,k-1)$ tangential. It remains to prove that

$$\phi[D \cap C_k(0,1/2)] > 0.$$

Since $0 \in D_j$ for all j we have $0 \in D$. Hence from (18),

$$\phi[C_k(0,1/2)] = 2^{-s}.$$

On the other hand, every point of $C_k(0,1/2) - D$ is contained in a sphere K such that

$$K \cap D_j \text{ is empty for all } j.$$

Hence this sphere contains a concentric sphere C such that

$$\phi C \leqq \liminf_{j\to\infty} \phi_j K = 0.$$

It now follows that

$$\phi[C_k(0,1/2) - D] = 0$$

and

$$\phi[D \cap C_k(0,1/2)] = 2^{-s}.$$

Consequently, $P(k,s,k-1)$ is false. This contradicts Lemma B and so Lemma 11 must be true, as required.

Proof of Lemma C. (Stated at the beginning of this section.) Let $0 \leqq s < k + 1 \leqq n$. We regard s and n as fixed, and use induction on k, starting with $k = n-1$, which is Lemma B.

We repeatedly apply Lemma 11, giving us the sequence of propositions

$$P(n,s,n-1),\ P(n,s,n-2),\cdots,P(n,s,t),$$

where $t+1$ is the least integer greater than s.

This completes the proof.

8. Lemma 12. *Let $\phi \in U_n''$, $s \geqq 0$, k a non-negative integer and let $B \subset \mathscr{B}_n$ be a weakly (ϕ,s,k) tangential set. Then $s \leqq k$.*

Proof. Choose a closed set $A \subset B$, where $\phi A > 0$, and $r > 0$, $l > 0$ such that at every point $x \in A$,

$$\frac{1}{2} l\rho^s < \phi[C_n(a,\rho)] < 2l\rho^s \text{ whenever } \rho < r. \tag{1}$$

Let $x_0 \in A$ be any weakly (ϕ,s,k) tangential point at which

$$\odot_n^s(\phi, E_n - A, x_0) = 0.$$

Then given any positive $\eta < 1$, we can find $r_0 < r$ and $R_0 \in \mathscr{G}_n$ such that

$$\phi[M_n^k(x_0, R_0, \eta r_0, r_0)] < \frac{1}{8} l r_0^s, \tag{2}$$

and

$$\phi[(E_n - A) \cap C_n(x_0, r_0)] < \frac{1}{8} l r_0^s. \tag{3}$$

Then, from (1) and (2),

$$\phi[L_n^k(x_0, R_0, \eta r_0, r_0)] > \frac{3}{8} l r_0^s,$$

and so from (3)

$$\phi[A \cap L_n^k(x_0, R_0, \eta r_0, r_0)] > \frac{1}{4} l r_0^s. \tag{4}$$

We shall obtain an inequality in the opposite direction by dividing $L_n^k(x_0, R_0, \eta r_0, r_0)$ into cubes and applying (1) to each of those cubes which intersect A. The cubes used are those bounded by the lines

$$x^{(j)} = m\eta r_0, \quad j = 1, \cdots, n,$$

where m takes integer values. The number of those cubes which intersect $L_n^k(x_0, R_0, \eta r_0, r_0)$ is $O(\eta^{-k})$.

Consider a typical cube C which contains a point $x \in A$. Then $C_n(x, 2^{n/2}\eta r_0) \supset C$, and hence from (1),

$$\phi(A \cap C) < 2^{(ns/2+1)} l \eta^s r_0^s.$$

Summing, we have

$$\phi[A \cap L_n^k(x_0, R_0, \eta r_0, r_0)] = O(l \eta^{s-k} r_0^s).$$

With (4), this gives us $\eta^{k-s} = O(1)$. Since η can be arbitrarily small, this is only possible with $s \leqq k$, as required.

9. **Proof of Theorem 2.** Trivially, B is weakly (ϕ, s, n) tangential and so by Lemma 12, $s \leqq n$.

Let k be the least integer greater than $s-1$, so that $s < k+1 \leqq n$, and, by Lemma C, $P(n, s, k)$ is true, whence B is weakly (ϕ, s, k) tangential. It follows from Lemma 12 that $s \leqq k$, which implies that $s = k$.

That is, (i) and (ii) of Theorem 2 are established as required.

Theorem 1 now follows by Lemma A.

References

1. A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*. II, Math. Ann. **115** (1938), 296–329.

2. H. Federer, *The (ϕ, k) rectifiable subsets of n space*, Trans. Amer. Math. Soc. **62** (1947), 114–192.

3. J. M. Marstrand, *Circular density of plane sets*, J. London Math. Soc. **30** (1955), 238–246.

4. ———, *Hausdorff* 2*-dimensional measure in* 3*-space*, Proc. London Math. Soc. (3) **11** (1961), 91–108.

5. E. F. Moore, *Density rations and* $(\phi, 1)$ *rectifiability in n-space*, Trans. Amer. Math. Soc. **69** (1950), 324–334.

6. A. P. Morse, *The role of internal families in measure theory*, Bull. Amer. Math. Soc. **50** (1944), 723–728.

7. A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. **55** (1944), 236–305.

8. S. Saks, *Theory of the integral*, Hafner, Warsaw, 1937; §9, p. 82.

Brown University,
Providence, Rhode Island